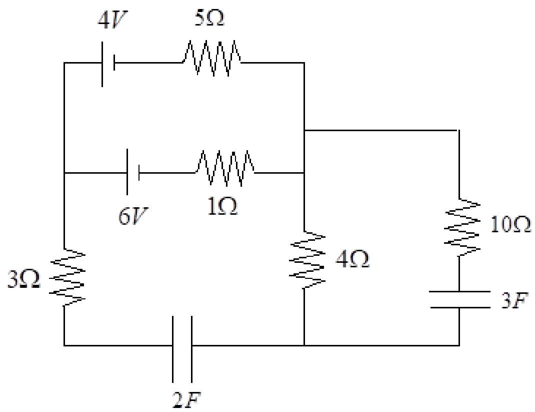
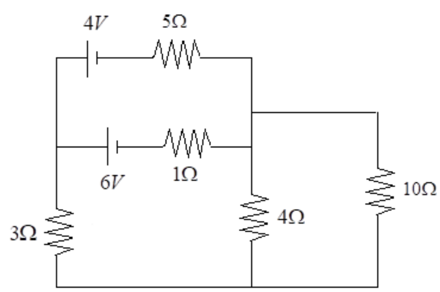
Homework 9 (Solutions): ArcySerkits

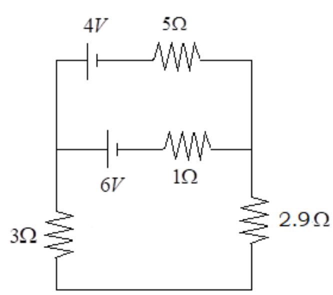
Problem 1. Suppose that initially uncharged capacitors are placed in this circuit. (a) What will be each's initial rate of charge (i.e. current)? (b) What will be each's final charge?



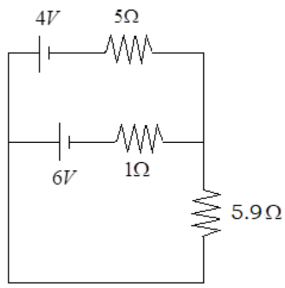
(a) To get the initial currents, we can short the capacitors (because they have no charge),



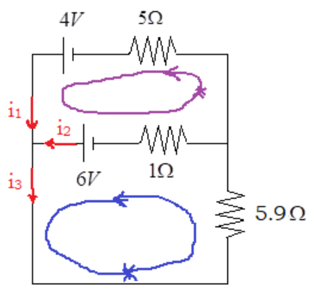
And combine the 4Ω, 10Ω to get $(4^{-1} + 10^{-1})^{-1} = 2.9\Omega$.



And then combine the 3Ω and 2.9Ω in series,



Then draw currents,



Applying KCL at the junction, and KVL around the loops,

$$-i_1 - i_2 + i_3 = 0$$
$$-5i_1 + 4 - 6 + 1i_2 = 0$$
$$-5.9i_3 - 1i_2 + 6 = 0$$

Simplifying a bit,

$$-i_1 - i_2 + i_3 = 0$$
$$-5i_1 + i_2 = 2$$
$$i_2 + 5.9i_3 = 6$$

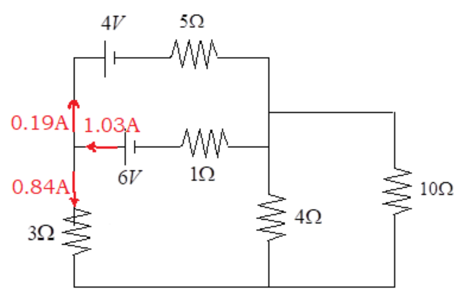
Doing the whole matrix thing,

$$\begin{pmatrix} -1 & -1 & 1 \\ -5 & 1 & 0 \\ 0 & 1 & 5.9 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$$

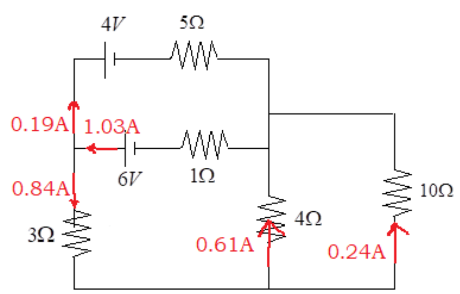
$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ -5 & 1 & 0 \\ 0 & 1 & 5.9 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -0.19 \\ 1.03 \\ 0.84 \end{pmatrix}$$

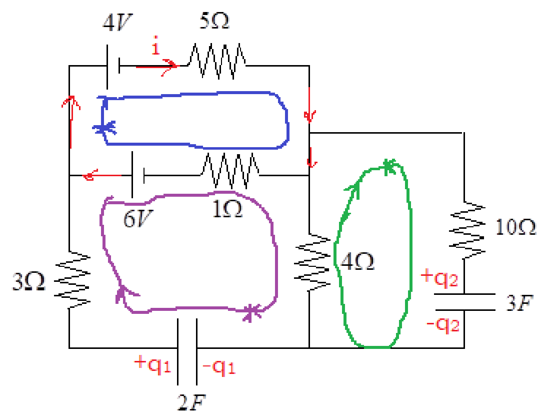
So, we have:



And the 0.84A current will split when it hits the junction running through the 4Ω and 10Ω resistors. The potential difference across these resistors will be (0.84A)(2.9Ω) = 2.44V, where 2.9Ω is their equivalent resistance. And so the current running through them will be: 2.44V/4Ω = 0.61A, and 0.24A. So we have:



(b) To get their final charge, we first recognize that no current will run through a wire with a fully charge capacitor in it. This will leave the current circulating around the top loop alone. Then we simply label the charges on the capacitors, and do KVL…



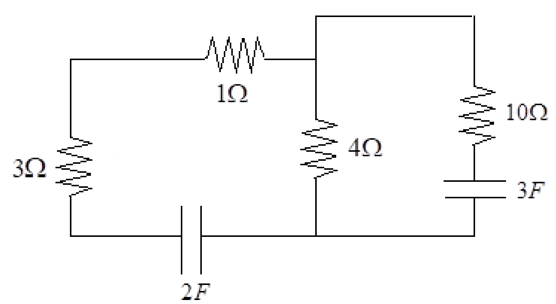
$$-4-5i-1i+6=0 \rightarrow 6i=2 \rightarrow i=0.33A$$

$$+\frac{q_1}{2}-(0A)(3\Omega)-6+(0.33A)(1\Omega)-(0A)(4\Omega)=0 \rightarrow q_1=11.33\text{C}$$

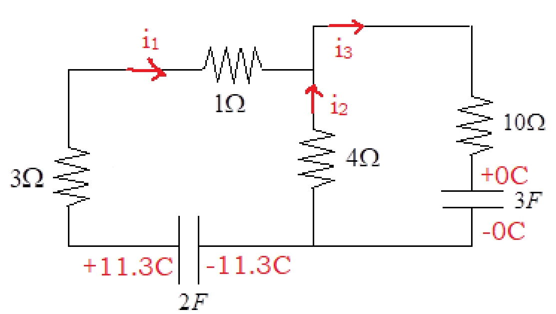
$$-(0A)(10\Omega)-\frac{q_2}{3}-(0A)(4\Omega)=0 \rightarrow q_2=0$$

So there we go: $i = 0.33A$, $q_1 = 11.33C$, $q_2 = 0C$.

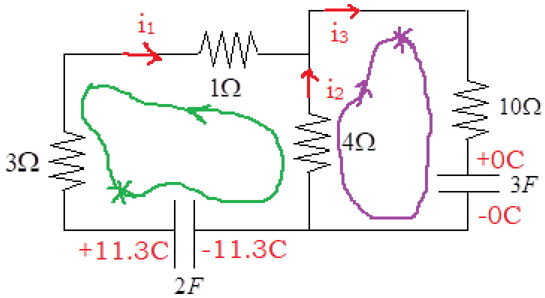
Problem 2. If the capacitors are then placed in the following circuit….(a) what will be their initial rates of discharge (currents again)? (b) What will be each's final charge? This should be equal to 1/(your self worth).



(a) So initial charges and currents are:



We could tackle this problem with equivalent resistance stuff, but I'll do KCL, KVL. So,



$$-i_1 - i_2 + i_3 = 0$$
$$-\frac{11.3}{2} - 4i_2 + 1i_1 + 3i_1 = 0$$
$$-10i_3 - \frac{0}{3} - 4i_2 = 0$$

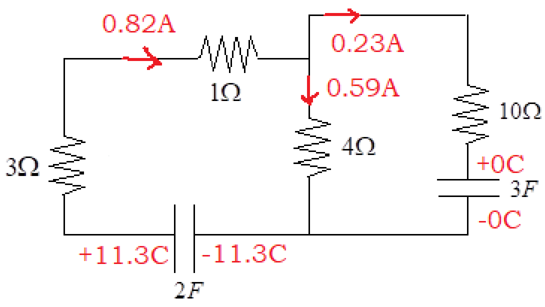
Fixin up,

$$-i_1 - i_2 + i_3 = 0$$
$$4i_1 - 4i_2 = 5.65$$
$$4i_2 + 10i_3 = 0$$

Matrix stuff,

$$\begin{pmatrix} -1 & -1 & 1 \\ 4 & -4 & 0 \\ 0 & 4 & 10 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5.65 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 4 & -4 & 0 \\ 0 & 4 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 5.65 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0.82 \\ -0.59 \\ 0.23 \end{pmatrix}$$

So here we are:

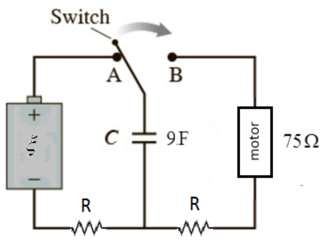


(b) Yep. All final charges are 1/∞ = 0.

Problem 3. Show that the dimensions of an Ohm·Farad work out to seconds.
Well,

$$\Omega \cdot F = \frac{\text{V}}{\text{A}} \cdot \frac{\text{C}}{\text{V}} = \frac{\text{C}}{\text{A}} = \frac{\text{C}}{\text{C/s}} = \text{s}$$

Problem 4. For each part, remember current will not run through an open break….



(a) Suppose that the 9F capacitor in the diagram is charged with an ξ = 150V battery, in series with an R = 25Ω resistor, while the switch is in position A. How long until the capacitor's potential difference reaches 120V?

An expression for the capacitor's potential difference is:

$$\begin{aligned} V(t) &= \frac{q(t)}{C} \\ &= \frac{q_\infty(1-e^{-t/\tau})}{C} \\ &= V_\infty(1-e^{-t/\tau}) \\ &= 150(1-e^{-t/225}) \end{aligned} \qquad \tau = RC = (25\Omega)(9F) = 225\text{s}$$

And we want to know when,

$$120 = 150(1 - e^{-t/225})$$
$$e^{-t/225} = 1 - \frac{120}{150}$$
$$\ln(e^{-t/225}) = \ln\left(1 - \frac{120}{150}\right)$$
$$-\frac{t}{225} = \ln\left(1 - \frac{120}{150}\right)$$
$$t = -225\ln\left(1 - \frac{120}{150}\right) = 362\text{s}$$

(b) Now suppose the motor can be approximated as a 75Ω resistor, and is in series with another 25Ω resistor. If the motor must have a potential difference of at least 30V across it to run, how long (in minutes) will it do so after the switch is flipped to position B?

Now we want the potential difference across the motor. This is:

$$\begin{aligned} V_{motor}(t) &= i(t)R \\ &= \left(-\frac{dq_{cap}(t)}{dt}\right)(75) \\ &= \left(-\frac{d}{dt}q_0 e^{-t/\tau}\right)(75) \qquad \tau = R_{eq.}C = (75\Omega + 25\Omega)(9F) = 900\text{s}, \quad q_0 = CV_0 = (9F)(120F) = 1080\,\text{C} \\ &= \left(-\frac{d}{dt}1080e^{-t/900}\right)(75) \\ &= \left(1080 \cdot \frac{1}{900}e^{-t/900}\right)(75) \\ &= 90e^{-t/900} \end{aligned}$$

And so want to know when this will drop below 30V. So,

$$30 = 90e^{-t/900}$$
$$\frac{1}{3} = e^{-t/900}$$
$$\ln(1/3) = -t/900$$
$$t = -900 \cdot \ln(1/3) = 989\text{s} \sim 16.5\,\text{min}$$